JANATA COLLEGE CHANDRAPUR

DEPARTMENT OF MATHEMATICS

SEM I (PAPER I)

TOPIC : BETA FUNCTION.

PRSEENTED BY

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Beta Function

Defination : The integral $\int_0^1 x^{m-1} (1-x)^{n-1}\, dx$ is a function of m,n , m>0, n>0 , & is called a Beta function. It is denoted by B(m,n).

Thus

$$B(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1}\, dx$$

Show that B(m,n)= $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$ = $\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$

→Proof :

We know that,

B(n,m)= B(m,n)= $\int_0^1 t^{m-1}(1-t)^{n-1}$ dt m,n>0 -- (1)

Now Put t= $\frac{x}{1+x}$ = 1- $\frac{1}{1+x}$

$$dt = \frac{dx}{(1+x)^2}$$

Taking

$$t = 0 ==> x = 0$$

$$t = 1 ==> x = \infty$$

From eq^n (1)

$\therefore$ B(n,m) = B(m,n) $= \int_0^\infty (1-\frac{1}{1+x})^{m-1}$ [1-$(1-\frac{1}{1+x})]^{n-1}$

$[\frac{1}{(1+x)^2}]dx$

$$= \int_0^\infty (\frac{x}{1+x})^{m-1} (\frac{1}{1+x})^{n-1} \left[\frac{1}{(1+x)^2}\right] dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m-1+n-1+2}} dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

Similarly

$$B(n,m) = B(m,n) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

Example : *Evaluate* $\int_0^\infty \frac{x^3}{(1+x)^7}dx$

$$\rightarrow \int_0^\infty \frac{x^3}{(1+x)^7}dx = \int_0^\infty \frac{x^{4-1}}{(1+x)^{4+3}}dx$$

$$= B(4,3)$$

$$= \frac{\Gamma 4 \, \Gamma 3}{\Gamma 4+3} \quad \text{------} \quad \left[B(m,n) = \frac{\Gamma m \, \Gamma n}{\Gamma m+n} \right]$$

$$= \frac{3!\,2!}{6!} = \frac{3\times2\times1\times2\times1}{6\times5\times4\times3\times2\times1} = \frac{1}{60}$$

2) show that B(m,n) = $\int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$

→ We have to show

$$B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx$$

We know

$$B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}} dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx$$

--- $\left[\int_0^\infty f(x)dx = \int_0^1 f(x)dx + \int_1^\infty f(x)dx \right]$

$$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{y^{m-1}}{(1+y)^{m+n}}\,dy \quad \text{---} \quad (1)$$

Now,

Put y = $\frac{1}{x}$ **in 2nd term**

$$\therefore\ dy = \frac{-1}{x^2}\,dx$$

Here y = 1 => x = 1

y = ∞ => x = 0

from (1)

$$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^0 \frac{\left(\frac{1}{x}\right)^{m-1}}{\left(1+\frac{1}{x}\right)^{m+n}}\left(\frac{-1}{x^2}\right)dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{m+n}}{x^{m-1}\,(x+1)^{m+n}}\left(\frac{1}{x^2}\right)dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{m+n-m+1-2}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\therefore \quad B(m,n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}$$

Thus proved ..


Thank You